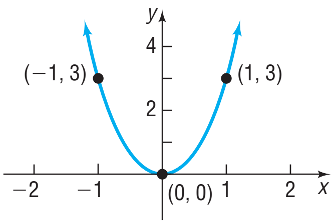
107. *Open the Amplitude applet.* On the screen you will see a slider. Move the point along the slider to see the role a plays in the graph of $f(x) = a \sin x$.

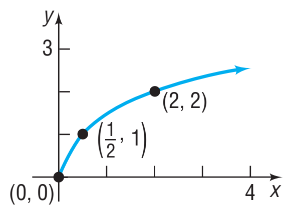
108. *Open the Period applet.* On the screen you will see a slider. Move the point along the slider to see the role ω plays in the graph of $f(x) = \sin(\omega x)$. Pay particular attention to the key points matched by color on each graph. For convenience the graph of $g(x) = \sin x$ is shown as a dashed, gray curve.

'Are You Prepared?' Answers

1. Vertical stretch by a factor of 3



2. Horizontal compression by a factor of $\frac{1}{2}$



7.7 Graphs of the Tangent, Cotangent, Cosecant, and Secant Functions

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Vertical Asymptotes (Section 5.2, pp. 345–346)

Now Work the **'Are You Prepared?'** problems on page 581.

OBJECTIVES
1. Graph Functions of the Form $y = A\tan(\omega x) + B$ and $y = A\cot(\omega x) + B$ (p. 578)
2. Graph Functions of the Form $y = A\csc(\omega x) + B$ and $y = A\sec(\omega x) + B$ (p. 580)

The Graph of the Tangent Function

Because the tangent function has period π, we only need to determine the graph over some interval of length π. The rest of the graph will consist of repetitions of that graph. Because the tangent function is not defined at $\dots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \dots$, we will concentrate on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, of length π, and construct Table 9, which lists some points on the graph of $y = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$. We plot the points in the table and connect them with a smooth curve. See Figure 95 for a partial graph of $y = \tan x$, where $-\frac{\pi}{3} \le x \le \frac{\pi}{3}$.

To complete one period of the graph of $y = \tan x$, we need to investigate the behavior of the function as x approaches $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. We must be careful, though, because $y = \tan x$ is not defined at these numbers. To determine this behavior, we use the identity

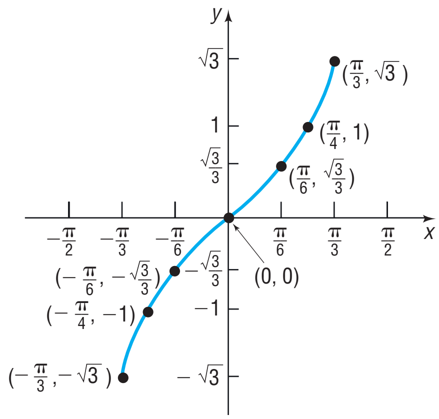
$$\tan x = \frac{\sin x}{\cos x}$$

See Table 10. If x is close to $\frac{\pi}{2} \approx 1.5708$, but remains less than $\frac{\pi}{2}$, then $\sin x$ will be close to 1 and $\cos x$ will be positive and close to 0. (To see this, refer back to the graphs of the sine function and the cosine function.) So the ratio $\frac{\sin x}{\cos x}$ will be

Table 9

x	$y = \tan x$	(x, y)
$-\frac{\pi}{3}$	$-\sqrt{3} \approx -1.73$	$\left(-\frac{\pi}{3}, -\sqrt{3}\right)$
$-\frac{\pi}{4}$	-1	$\left(-\frac{\pi}{4}, -1\right)$
$-\frac{\pi}{6}$	$-\frac{\sqrt{3}}{3} \approx -0.58$	$\left(-\frac{\pi}{6}, -\frac{\sqrt{3}}{3}\right)$
0	0	(0, 0)
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{3} \approx 0.58$	$\left(\frac{\pi}{6}, \frac{\sqrt{3}}{3}\right)$
$\frac{\pi}{4}$	1	$\left(\frac{\pi}{4}, 1\right)$
$\frac{\pi}{3}$	$\sqrt{3} \approx 1.73$	$\left(\frac{\pi}{3}, \sqrt{3}\right)$

Figure 95
$y = \tan x, -\frac{\pi}{3} \le x \le \frac{\pi}{3}$



positive and large. In fact, the closer x gets to $\frac{\pi}{2}$, the closer $\sin x$ gets to 1 and $\cos x$ gets to 0, so $\tan x$ approaches ∞ $\left(\lim_{x \to \frac{\pi}{2}^-} \tan x = \infty\right)$. In other words, the vertical line $x = \frac{\pi}{2}$ is a vertical asymptote to the graph of $y = \tan x$.

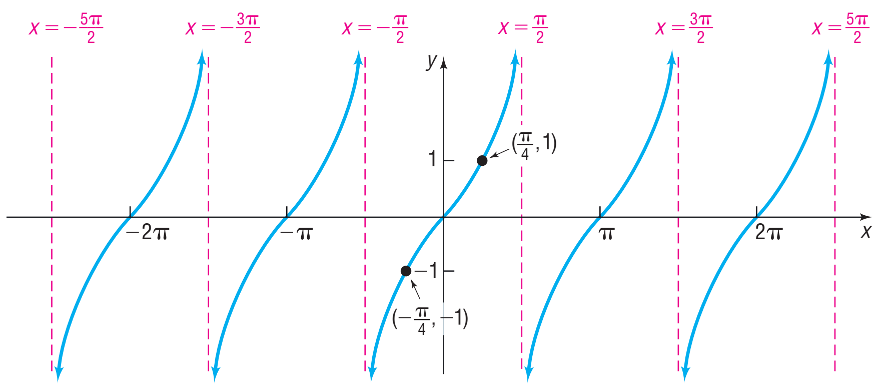
Table 10

x	$\sin x$	$\cos x$	$y = \tan x$
$\frac{\pi}{3} \approx 1.05$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3} \approx 1.73$
1.5	0.9975	0.0707	14.1
1.57	0.9999	7.96×10^{-4}	1255.8
1.5707	0.9999	9.6×10^{-5}	10,381
$\frac{\pi}{2} \approx 1.5708$	1	0	Undefined

If x is close to $-\frac{\pi}{2}$, but remains greater than $-\frac{\pi}{2}$, then $\sin x$ will be close to -1 and $\cos x$ will be positive and close to 0. The ratio $\frac{\sin x}{\cos x}$ approaches $-\infty$ $\left(\lim_{x \to -\frac{\pi}{2}^+} \tan x = -\infty\right)$. In other words, the vertical line $x = -\frac{\pi}{2}$ is also a vertical asymptote to the graph.

With these observations, we can complete one period of the graph. We obtain the complete graph of $y = \tan x$ by repeating this period, as shown in Figure 96.

Figure 96
$y = \tan x, -\infty < x < \infty$, x not equal to odd multiples of $\frac{\pi}{2}$, $-\infty < y < \infty$



Check: Graph $Y_1 = \tan x$ and compare the result with Figure 96. Use TRACE to see what happens as x gets close to $\frac{\pi}{2}$, but is less than $\frac{\pi}{2}$.

The graph of $y = \tan x$ in Figure 96 on page 577 illustrates the following properties.

Properties of the Tangent Function

1. The domain is the set of all real numbers, except odd multiples of $\frac{\pi}{2}$.
2. The range is the set of all real numbers.
3. The tangent function is an odd function, as the symmetry of the graph with respect to the origin indicates.
4. The tangent function is periodic, with period π.
5. The x-intercepts are $\ldots, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$; the y-intercept is 0.
6. Vertical asymptotes occur at $x = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$.

Now Work PROBLEMS 7 AND 15

1 Graph Functions of the Form $y = A\tan(\omega x) + B$ and $y = A\cot(\omega x) + B$

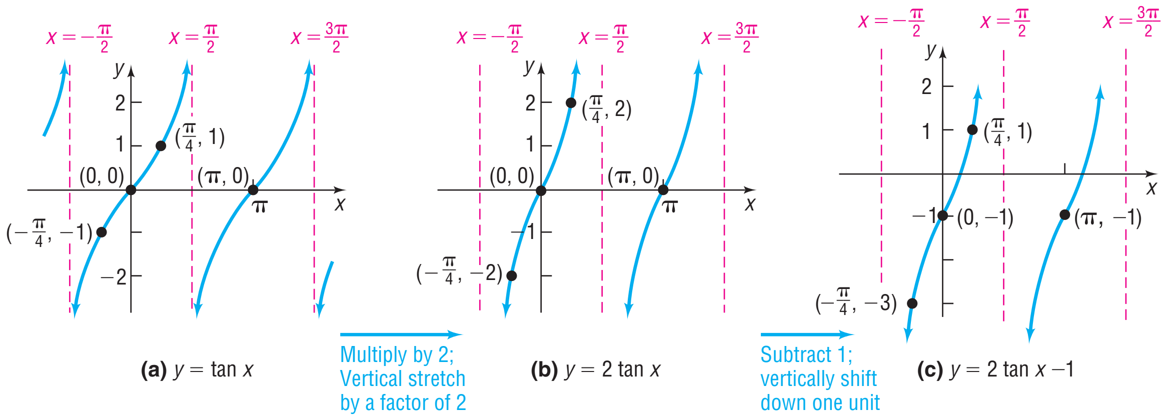
For tangent functions, there is no concept of amplitude since the range of the tangent function is $(-\infty, \infty)$. The role of A in $y = A\tan(\omega x) + B$ is to provide the magnitude of the vertical stretch. The period of $y = \tan x$ is π, so the period of $y = A\tan(\omega x) + B$ is $\frac{\pi}{\omega}$, caused by the horizontal compression of the graph by a factor of $\frac{1}{\omega}$. Finally, the presence of B indicates that a vertical shift is required.

EXAMPLE 1 **Graphing Functions of the Form $y = A\tan(\omega x) + B$**

Graph: $y = 2\tan x - 1$. Use the graph to determine the domain and the range of $y = 2\tan x - 1$.

Solution Figure 97 shows the steps using transformations.

Figure 97



✓Check: Graph $Y_1 = 2\tan x - 1$ to verify the graph shown in Figure 97(c).

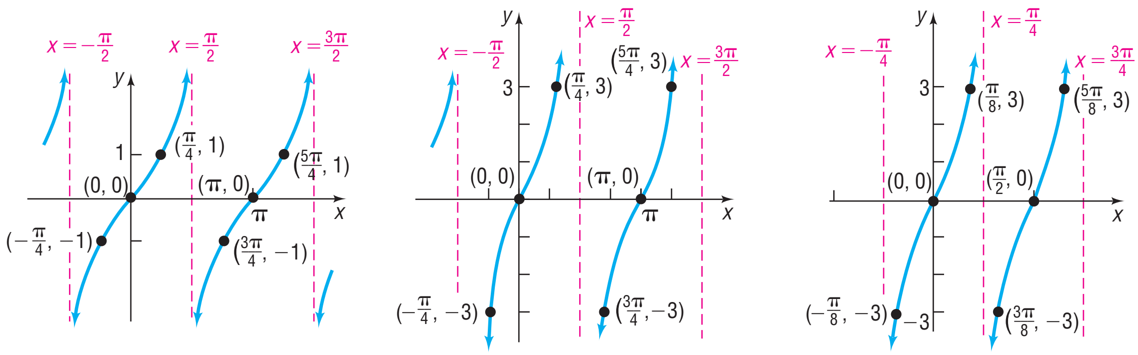
The domain of $y = 2\tan x - 1$ is $\left\{x \middle| x \neq \frac{k\pi}{2}, k \text{ is an odd integer}\right\}$, and the range is the set of all real numbers, or $(-\infty, \infty)$.

EXAMPLE 2 **Graphing Functions of the Form $y = A\tan(\omega x) + B$**

Graph $y = 3\tan(2x)$. Use the graph to determine the domain and the range of $y = 3\tan(2x)$.

Solution Figure 98 shows the steps using transformations.

Figure 98



(a) $y = \tan x$ (b) $y = 3\tan x$ (c) $y = 3\tan(2x)$

The domain of $y = 3\tan(2x)$ is $\left\{x \,\middle|\, x \neq \frac{k\pi}{4}, k \text{ is an odd integer}\right\}$, and the range is the set of all real numbers or $(-\infty, \infty)$.

Check: Graph $Y_1 = 3\tan(2x)$ to verify the graph in Figure 98(c).

Notice in Figure 98(c) that the period of $y = 3\tan(2x)$ is $\frac{\pi}{2}$ due to the compression of the original period π by a factor of $\frac{1}{2}$. Notice that the asymptotes are $x = -\frac{\pi}{4}$, $x = \frac{\pi}{4}$, $x = \frac{3\pi}{4}$, and so on, also due to the compression.

Now Work PROBLEM 21

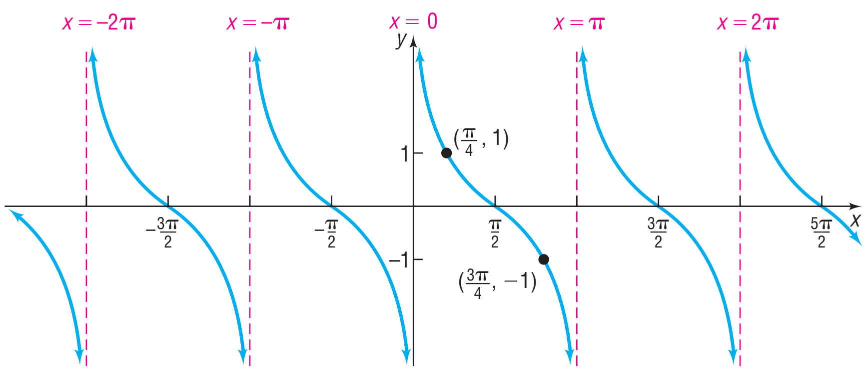
The Graph of the Cotangent Function

We obtain the graph of $y = \cot x$ as we did the graph of $y = \tan x$. The period of $y = \cot x$ is π. Because the cotangent function is not defined for integer multiples of π, we will concentrate on the interval $(0, \pi)$. Table 11 lists some points on the graph of $y = \cot x$, $0 < x < \pi$. As x approaches 0, but remains greater than 0, the value of $\cos x$ will be close to 1 and the value of $\sin x$ will be positive and close to 0. Hence, the ratio $\frac{\cos x}{\sin x} = \cot x$ will be positive and large; so as x approaches 0, with $x > 0$, $\cot x$ approaches $\infty \left(\lim_{x \to 0^+} \cot x = \infty\right)$. Similarly, as x approaches π, but remains less than π, the value of $\cos x$ will be close to -1, and the value of $\sin x$ will be positive and close to 0. So the ratio $\frac{\cos x}{\sin x} = \cot x$ will be negative and will approach $-\infty$ as x approaches $\pi \left(\lim_{x \to \pi^-} \cot x = -\infty\right)$. Figure 99 shows the graph.

Table 11

x	$y = \cot x$	(x, y)
$\frac{\pi}{6}$	$\sqrt{3}$	$\left(\frac{\pi}{6}, \sqrt{3}\right)$
$\frac{\pi}{4}$	1	$\left(\frac{\pi}{4}, 1\right)$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{3}$	$\left(\frac{\pi}{3}, \frac{\sqrt{3}}{3}\right)$
$\frac{\pi}{2}$	0	$\left(\frac{\pi}{2}, 0\right)$
$\frac{2\pi}{3}$	$-\frac{\sqrt{3}}{3}$	$\left(\frac{2\pi}{3}, -\frac{\sqrt{3}}{3}\right)$
$\frac{3\pi}{4}$	-1	$\left(\frac{3\pi}{4}, -1\right)$
$\frac{5\pi}{6}$	$-\sqrt{3}$	$\left(\frac{5\pi}{6}, -\sqrt{3}\right)$

Figure 99
$y = \cot x$, $-\infty < x < \infty$, x not equal to integer multiples of π, $-\infty < y < \infty$



The graph of $y = A\cot(\omega x) + B$ has similar characteristics to those of the tangent function. The cotangent function $y = A\cot(\omega x) + B$ has period $\frac{\pi}{\omega}$. The cotangent function has no amplitude. The role of A is to provide the magnitude of the vertical stretch; the presence of B indicates a vertical shift is required.

Now Work PROBLEM 23

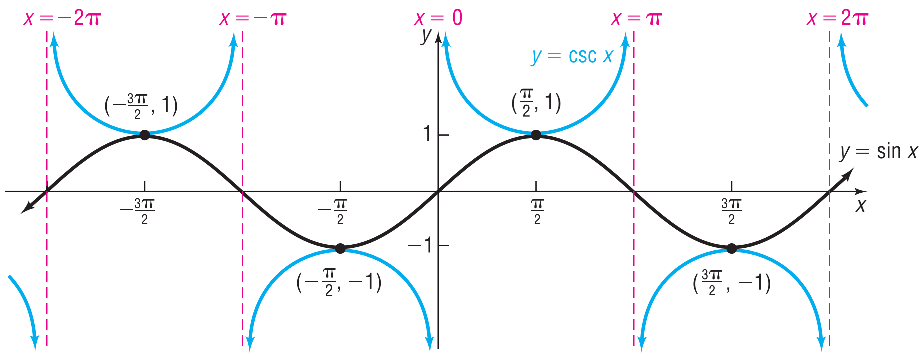
The Graphs of the Cosecant Function and the Secant Function

The cosecant and secant functions, sometimes referred to as **reciprocal functions,** are graphed by making use of the reciprocal identities

$$\csc x = \frac{1}{\sin x} \quad \text{and} \quad \sec x = \frac{1}{\cos x}$$

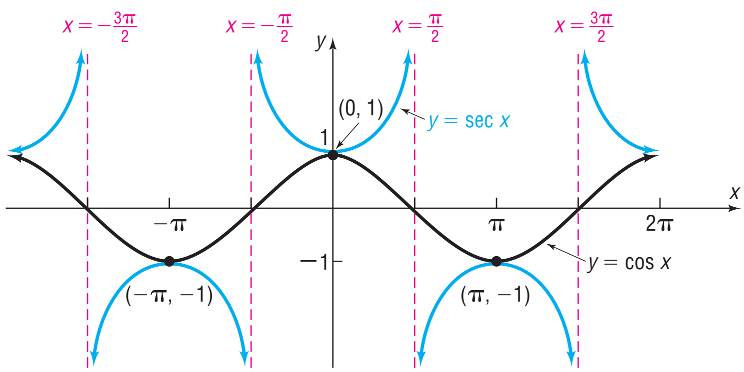
For example, the value of the cosecant function $y = \csc x$ at a given number x equals the reciprocal of the corresponding value of the sine function, provided that the value of the sine function is not 0. If the value of $\sin x$ is 0, then x is an integer multiple of π. At such numbers, the cosecant function is not defined. In fact, the graph of the cosecant function has vertical asymptotes at integer multiples of π. Figure 100 shows the graph.

Figure 100
$y = \csc x, -\infty < x < \infty, x$ not equal to integer multiples of π, $|y| \geq 1$



Using the idea of reciprocals, we can similarly obtain the graph of $y = \sec x$. See Figure 101.

Figure 101
$y = \sec x, -\infty < x < \infty, x$ not equal to odd multiples of $\frac{\pi}{2}$, $|y| \geq 1$



2 Graph Functions of the Form $y = A\csc(\omega x) + B$ and $y = A\sec(\omega x) + B$

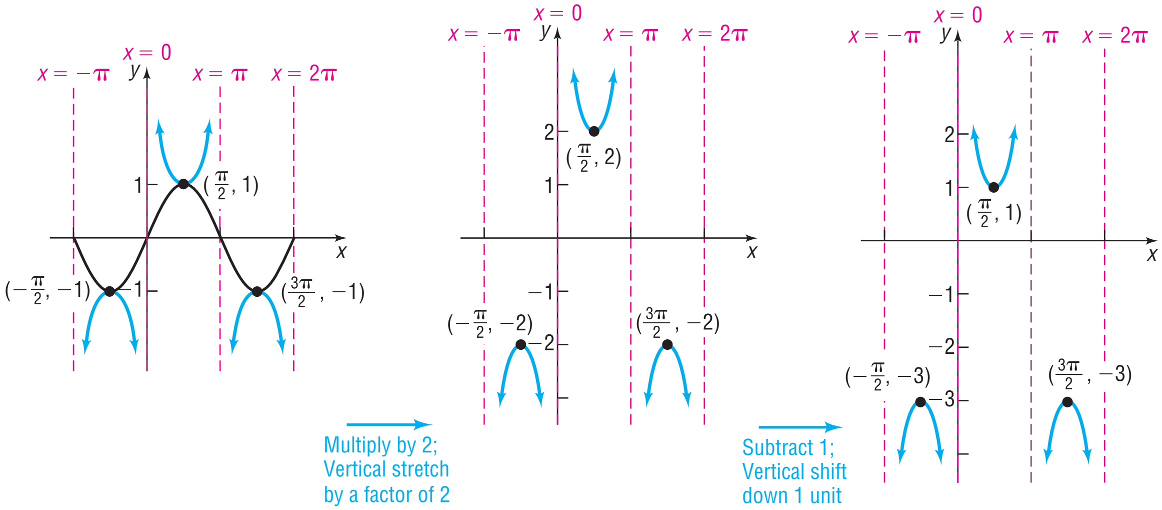
The role of A in these functions is to set the range. The range of $y = \csc x$ is $\{y | y \leq -1 \text{ or } y \geq 1\}$ or $\{y | |y| \geq 1\}$; the range of $y = A\csc x$ is $\{y | |y| \geq |A|\}$, due to the vertical stretch of the graph by a factor of $|A|$. Just as with the sine and cosine functions, the period of $y = \csc(\omega x)$ and $y = \sec(\omega x)$ becomes $\frac{2\pi}{\omega}$, due to the horizontal compression of the graph by a factor of $\frac{1}{\omega}$. The presence of B indicates that a vertical shift is required.

EXAMPLE 3 Graphing Functions of the Form $y = A\csc(\omega x) + B$

Graph $y = 2\csc x - 1$. Use the graph to determine the domain and the range of $y = 2\csc x - 1$.

Solution We use transformations. Figure 102 shows the required steps.

Figure 102



(a) $y = \csc x$ **(b)** $y = 2\csc x$ **(c)** $y = 2\csc x - 1$

The domain of $y = 2\csc x - 1$ is $\{x | x \neq k\pi, k \text{ is an integer}\}$ and the range is $\{y | y \leq -3 \text{ or } y \geq 1\}$ or, using interval notation, $(-\infty, -3] \cup [1, \infty)$.

Check: Graph $Y_1 = 2\csc x - 1$ to verify the graph shown in Figure 102.

Now Work PROBLEM 29

7.7 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. The graph of $y = \frac{3x - 6}{x - 4}$ has a vertical asymptote. What is it? (pp. 345–346)

2. ***True or False*** If $x = 3$ is a vertical asymptote of a rational function R, then $\lim_{x \to 3} |R(x)| = \infty$. (pp. 345–346)

Concepts and Vocabulary

3. The graph of $y = \tan x$ is symmetric with respect to the _______ and has vertical asymptotes at ________________ .

4. The graph of $y = \sec x$ is symmetric with respect to the _______ and has vertical asymptotes at ________________ .

5. It is easiest to graph $y = \sec x$ by first sketching the graph of ____________.

6. ***True or False*** The graphs of $y = \tan x$, $y = \cot x$, $y = \sec x$, and $y = \csc x$ each have infinitely many vertical asymptotes.

Skill Building

In Problems 7–16, if necessary, refer to the graphs to answer each question.

7. What is the y-intercept of $y = \tan x$?

8. What is the y-intercept of $y = \cot x$?

9. What is the y-intercept of $y = \sec x$?

10. What is the y-intercept of $y = \csc x$?

11. For what numbers x, $-2\pi \le x \le 2\pi$, does $\sec x = 1$? For what numbers x does $\sec x = -1$?

12. For what numbers x, $-2\pi \le x \le 2\pi$, does $\csc x = 1$? For what numbers x does $\csc x = -1$?

13. For what numbers x, $-2\pi \le x \le 2\pi$, does the graph of $y = \sec x$ have vertical asymptotes?

14. For what numbers x, $-2\pi \le x \le 2\pi$, does the graph of $y = \csc x$ have vertical asymptotes?

15. For what numbers x, $-2\pi \le x \le 2\pi$, does the graph of $y = \tan x$ have vertical asymptotes?

16. For what numbers x, $-2\pi \le x \le 2\pi$, does the graph of $y = \cot x$ have vertical asymptotes?

In Problems 17–40, graph each function. Be sure to label key points and show at least two cycles. Use the graph to determine the domain and the range of each function.

17. $y = 3\tan x$

18. $y = -2\tan x$

19. $y = 4\cot x$

20. $y = -3\cot x$

21. $y = \tan\left(\frac{\pi}{2}x\right)$

22. $y = \tan\left(\frac{1}{2}x\right)$

23. $y = \cot\left(\frac{1}{4}x\right)$

24. $y = \cot\left(\frac{\pi}{4}x\right)$

25. $y = 2\sec x$

26. $y = \frac{1}{2}\csc x$

27. $y = -3\csc x$

28. $y = -4\sec x$

29. $y = 4\sec\left(\frac{1}{2}x\right)$

30. $y = \frac{1}{2}\csc(2x)$

31. $y = -2\csc(\pi x)$

32. $y = -3\sec\left(\frac{\pi}{2}x\right)$

33. $y = \tan\left(\frac{1}{4}x\right) + 1$

34. $y = 2\cot x - 1$

35. $y = \sec\left(\frac{2\pi}{3}x\right) + 2$

36. $y = \csc\left(\frac{3\pi}{2}x\right)$

37. $y = \frac{1}{2}\tan\left(\frac{1}{4}x\right) - 2$

38. $y = 3\cot\left(\frac{1}{2}x\right) - 2$

39. $y = 2\csc\left(\frac{1}{3}x\right) - 1$

40. $y = 3\sec\left(\frac{1}{4}x\right) + 1$

Mixed Practice

In Problems 41–44, find the average rate of change of f from 0 to $\frac{\pi}{6}$.

41. $f(x) = \tan x$

42. $f(x) = \sec x$

43. $f(x) = \tan(2x)$

44. $f(x) = \sec(2x)$

In Problems 45–48, find $(f \circ g)(x)$ *and* $(g \circ f)(x)$ *and graph each of these functions.*

45. $f(x) = \tan x$
$g(x) = 4x$

46. $f(x) = 2\sec x$
$g(x) = \frac{1}{2}x$

47. $f(x) = -2x$
$g(x) = \cot x$

48. $f(x) = \frac{1}{2}x$
$g(x) = 2\csc x$

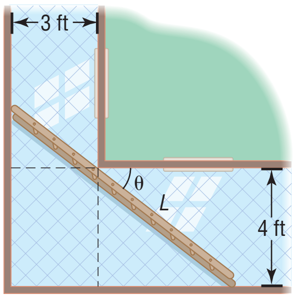
In Problems 49 and 50, graph each function.

49. $$f(x) = \begin{cases} \tan x & 0 \le x < \frac{\pi}{2} \\ 0 & x = \frac{\pi}{2} \\ \sec x & \frac{\pi}{2} < x \le \pi \end{cases}$$

50. $$g(x) = \begin{cases} \csc x & 0 < x < \pi \\ 0 & x = \pi \\ \cot x & \pi < x < 2\pi \end{cases}$$

Applications and Extensions

51. Carrying a Ladder around a Corner Two hallways, one of width 3 feet, the other of width 4 feet, meet at a right angle. See the illustration.



(a) Show that the length L of the line segment shown as a function of the angle θ is

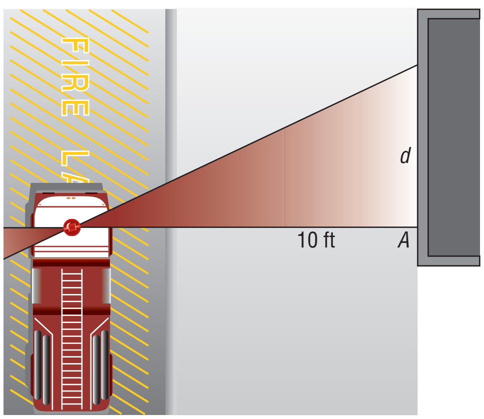
$$L(\theta) = 3 \sec \theta + 4 \csc \theta$$

(b) Graph $L = L(\theta), 0 < \theta < \frac{\pi}{2}$.

(c) For what value of θ is L the least?

(d) What is the length of the longest ladder that can be carried around the corner? Why is this also the least value of L?

52. A Rotating Beacon Suppose that a fire truck is parked in front of a building as shown in the figure.



The beacon light on top of the fire truck is located 10 feet from the wall and has a light on each side. If the beacon light rotates 1 revolution every 2 seconds, then a model for determining the distance d, in feet, that the beacon of light is from point A on the wall after t seconds is given by

$$d(t) = |10 \tan(\pi t)|$$

(a) Graph $d(t) = |10 \tan(\pi t)|$ for $0 \le t \le 2$.

(b) For what values of t is the function undefined? Explain what this means in terms of the beam of light on the wall.

(c) Fill in the following table.

t	0	0.1	0.2	0.3	0.4
$d(t) = 10 \tan(\pi t)$					

(d) Compute $\dfrac{d(0.1) - d(0)}{0.1 - 0}, \dfrac{d(0.2) - d(0.1)}{0.2 - 0.1}$, and so on, for each consecutive value of t. These are called **first differences.**

(e) Interpret the first differences found in part (d). What is happening to the speed of the beam of light as d increases?

53. Exploration Graph

$$y = \tan x \quad \text{and} \quad y = -\cot\left(x + \frac{\pi}{2}\right)$$

Do you think that $\tan x = -\cot\left(x + \dfrac{\pi}{2}\right)$?

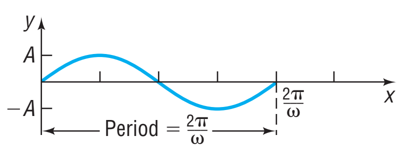
'Are You Prepared?' Answers

1. $x = 4$ **2.** True

7.8 Phase Shift; Sinusoidal Curve Fitting

OBJECTIVES
1 Graph Sinusoidal Functions of the Form $y = A \sin(\omega x - \phi) + B$ (p. 583)
2 Build Sinusoidal Models from Data (p. 587)

Figure 103 One cycle of $y = A \sin(\omega x), A > 0, \omega > 0$



1 Graph Sinusoidal Functions of the Form $y = A \sin(\omega x - \phi) + B$

We have seen that the graph of $y = A \sin(\omega x), \omega > 0$, has amplitude $|A|$ and period $T = \dfrac{2\pi}{\omega}$. One cycle can be drawn as x varies from 0 to $\dfrac{2\pi}{\omega}$ or, equivalently, as ωx varies from 0 to 2π. See Figure 103.